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Non-Commutative Algebras and Quantum Structures

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We present a survey on pseudo-effect algebras and pseudo MV-algebras, which generalize effect algebras and MV-algebras by dropping the assumption on commutativity. A non-commutative logic is nowadays used even in programming languages. We show when a pseudo-effect algebra E is an interval in a unital po-group. This is possible, e.g. if E satisfies a Riesz-type decomposition property, i.e. another kind of distributivity with respect to addition. Every pseudo MV-algebra is an interval in a unital ℓ-group. We study a case when compatibility can be expressed by a pseudo MV-structure, i.e. when E can be covered by blocks being pseudo MV-algebras. Finally, we study the state space of such structures.

KEY WORDS: pseudo-effect algebra; pseudo MV-algebras; po-group; unital po-group; unital ℓ-group; compatibility; block; state; extremal state.

1. INTRODUCTION

Recently we have commemorated 100 years since the Second International Congress of Mathematics, which was held in Paris, 1900, and at which D. Hilbert addressed his historical lecture on open mathematical problems. His program substantially influenced the development of mathematics in the 20th century, and also nowadays, on the doorstep to the third millennium, we can say that Hilbert's program is still influencing and many new generations of mathematicians will have to do with this program.

His sixth problem is lying among mathematics and physics and concerns mathematical foundations of quantum mechanics and it says: *Find a few physical axioms which, similar to the axioms of geometry, can describe a theory for a class of physical events that is as large as possible*.

The development of this problem had many highlights. The situation in physics and in mathematics was very interesting in 1930s. Kolmogorov published in 1933 his fundamental work *Grundbegriffe der Wahscheinlichkeitsrechnung*

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(Kolmogorov, 1933), where he presented axiomatical foundations of probability theory. This model is based on a σ-algebra of subsets, or a Boolean σ-algebra, which represents events describing a system under study. But as it follows, e.g. from the Heisenberg uncertainty principle, this model does not describe a propositional system of quantum mechanics.

In 1936, Birkhoff and von Neumann published their influencing article *The logic of quantum mechanics* (Birkhoff and von Neumann, 1936), where they propose a more general propositional structure for describing the events of quantum mechanics. Since that date we have fixed a birth of a new theory, quantum logics theory.

The boom of quantum logics started in 1960s after the fundamental paper of Gleason published in 1957, describing states on the most important quantum logic, the logic $\mathcal{L}(H)$, the system of all closed subspaces of a Hilbert space H, and after the first monograph on this topics published by Varadarajan (1968).

After that, there appeared a whole system of structures dedicated to mathematical foundations of quantum mechanics: *orthomodular lattices, orthomodular posets, σ-quantum logics*, etc. and also *orthoalgebras* by Foulis and Randall (1972). All these structures have an important property that they try to describe a way how to combine new events, disjunction of two propositions. They suggest to do so only when these are mutually exclusive. In addition, the second important feature is that they describe only no–yes events, i.e. using only two-valued logic.

A dramatic moment appeared in the beginning of 1990s when two former students of the first author, Kôpka and Chovanec (1994), introduced a new algebraic structure, *difference posets* or, abbreviated, *D-posets*, which reflects both algebraic and fuzzy ideas for a propositional system. Their primary notion was a difference of two comparative events. Then Foulis and Bennett (1993) gave an equivalent partial algebraic structure, *effect algebras*, with addition of mutually exclusive events as a primary notion. It was recognized that these former structures are both equivalent to the weak orthoalgebras introduced by Giuntini and Greuling (1989). The most important example is the system $\mathcal{E}(H)$ of effects of a Hilbert space, i.e. of all Hermitian operators among the null and identity operator. The second important example of effect algebras are MV-algebras which entered mathematics in 1950s by Chang (1958) as a many-valued reasoning. MV-algebras correspond to Boolean algebras in the framework of effect algebras, and they describe a "fuzzy classical situation" because recently Riečanová (2000) showed that every lattice-ordered effect algebra can be covered by blocks, maximal systems of compatible elements, and these blocks are MV-subalgebras.

Another important feature of effect algebras is that they appear as intervals in partially ordered groups with strong units.[3] For example, $\mathcal{E}(H)$ is an interval

[3] A positive element u of a po-group G is said to be a *strong unit* for G if, for any $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

of the po-group $\mathcal{B}(H)$, the system of all Hermitian operators of a Hilbert space H with the strong unit I, the identity operator of H.

Going back to those quantum structures, we see that there appeared an original idea of Boole (1967), who many decades before said that what we need to measure, are only pairs of events which are roughly speaking "mutually exclusive."

Today it is clear that phenomena similar to quantum mechanics, i.e. situations when it is not possible to use Kolmogorov probability models, do not only appear in quantum mechanics. Similar events we can observe in psychology, in the work of the human brain (Stern, 1994), in big computer systems or in economical systems (Dvurečenskij and Graziano, in press).

All mentioned quantum structures, nevertheless their "non-commutativity", they are commutative, i.e. the partial operation $+$ is commutative, i.e. $a + b = b + a$. However, a non-commutative reasoning can be met in the everyday life very often. Many human processes are depending on the order of variables: In clinical medicine, on behalf of the transplantation of human organs, an experiment was performed in which the same two questions have been posed to two groups of interviewed people: (1) Do you agree to dedicate your organs for medical transplantation after your death? (2) Do you agree to accept organs of a donor in the case of your need? When the order of questions was changed in the second group, the number of positive answers to be a donor here was more higher than for the first group.

Today there exists even a programming language (Baudot, 2000) based on a non-commutative logic.

Clearly also quantum mechanical measurements are in general non-commutative; the result of some experiment may depend on the order of the measurements. Consider, for example, a beam of particles which are prepared in a certain state, and which are sent through a sequence of three polarizing filters F_1, F_2, F_3. It is well known that the order of the filters makes in general a difference. For example, let the filter be polarizing in planes perpendicular to the particle beam, such that F_1 polarizes vertically, F_2 horizontally and F_3 at a 45° angle. If we place the filters in the order F_1, F_2, F_3, then no particles are detected, but in the order F_1, F_3, F_2, particles are detected; the difference is due to quantum interference.

Such phenomena are in literature also presented as sequential conjunctions or sequentially independent effects by Gudder and Nagy (2002) or sequential probability models by Foulis (2002).

Nowadays a whole family of non-commutative generalizations of MV-algebras have appeared: pseudo MV-algebras of Georgescu and Iorgulascu (2001), or equivalently, generalized MV-algebras of Rachůnek (2002), pseudo BL-algebras (Di Nola *et al.*, 2002). For them the author (Dvurečenskij, 2002) proved that any pseudo MV-algebra is always an interval in a unital ℓ-group (G, u) with a strong unit u. In addition, pseudo-effect algebras were introduced in (Dvurečenskij and Vetterlein, 2001a,b). Such algebras are sometimes also intervals in unital po-groups.

The aim of the present paper is to show new ideas using this kind of non-commutativity. The paper is organized as follows. In the second section, we describe pseudo-effect algebras and we show when a pseudo-effect describes a whole po-group, in which it is an interval. In the third section, we give elements of pseudo MV-algebras theory; such algebras are always intervals in unital ℓ-groups. In the fourth section, we show how compatibility is connected with blocks and these with pseudo MV-algebras.

The state space of pseudo MV-algebras and pseudo-effect algebras are presented in the fifth section.

2. PSEUDO-EFFECT ALGEBRAS

A partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* (Dvurečenskij and Vetterlein, 2001a,b) if, for all $a, b, c \in E$, the following holds

(i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$.
(ii) for any $a \in E$, there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;
(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. If E is a lattice under $\leq$, we say that E is a *lattice pseudo-effect algebra*. If $+$ is commutative, i.e. if $a + b = b + a$, for all $a, b \in E$ such that $b + a \in E$, E is said to be an *effect algebra*.

Let E be a pseudo-effect algebra. Let $/$, $\setminus$ be two partial binary operations on E such that, for $a, b \in E$, a / b is defined iff $b \setminus a$ is defined iff $a \leq b$, and such that in this case we have

$$(b \setminus a) + a = a + (a / b) = b. \tag{1}$$

Then

$$a = (b \setminus a) / b = b \setminus (a / b). \tag{2}$$

If $a \leq b \leq c$, then

$$(c \setminus a) \setminus (b \setminus a) = c \setminus b,$$

$$(a / b) / (a / c) = b / c,$$

$$(c \setminus b) / (c \setminus a) = b \setminus a,$$

$$(a / c) \setminus (b / c) = a / b.$$

Let $E = (E; +, 0, 1)$ be a pseudo-effect algebra. We define $a^- := 1 \setminus a$ and $a^\sim := a \, / \, 1$ for any $a \in E$.

For example if (G, u) is a unital (not necessary Abelian) po-group with a strong unit u (sometimes it is sufficient to assume only $u > 0$), and

$$\Gamma(G, u) := [0, u] = \{g \in G : \ 0 \leq g \leq u\},$$

then $(\Gamma(G, u); +, 0, u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$. A pseudo-effect algebra $(E; +, 0, 1)$ is said to be an *interval pseudo-effect algebra* if there exists a unital po-group (G, u) such that $(E; +, 0, 1)$ is isomorphic with $(\Gamma(G, u); +, 0, u)$.

For example, $\mathcal{E}(H) = \Gamma(\mathcal{B}(H), I)$ is an interval effect algebra.

The conditions when a pseudo-effect algebra is an interval were studied in (Dvurečenskij and Vetterlein, 2001a,b). For them we need the following generalizations of the Riesz decomposition property:

(a) For $a, b \in E$, we write a **com** b to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz interpolation property* (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \ \leq \ b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \ \leq \ c \ \leq \ b_1, b_2$.

(c) We say that E fulfils the *weak Riesz decomposition property* (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1, \ d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Riesz decomposition property* (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1, \ d_3 + d_4 = a_2, \ d_1 + d_3 = b_1, \ d_2 + d_4 = b_2$.

(e) We say that E fulfils the *commutational Riesz decomposition property* (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1, \ d_3 + d_4 = a_2, \ d_1 + d_3 = b_1, \ d_2 + d_4 = b_2$, and (ii) d_2 **com** d_3.

(f) We say that E fulfils the *strong Riesz decomposition property* (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1, \ d_3 + d_4 = a_2, \ d_1 + d_3 = b_1, \ d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

We notify that the Riesz-type decomposition properties are roughly speaking another kind of distributivity; they are connected with the refinements of decompositions of the unity.

We have the implications

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Rightarrow (\text{RDP}) \Rightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}),$$

and the opposite ones can fail.

If the positive cone G^+ of a po-group G satisfies analogical conditions, we say that G satisfies the corresponding type of the Riesz decomposition property. The following important representation theorem for pseudo-effect algebras says that every pseudo-effect algebra with (RDP_1) is an interval.

Theorem 2.1. (Dvurečenskij and Vetterlein, 2001b) *Let* $(E; +, 0, 1)$ *be a pseudo-effect algebra fulfilling* (RDP_1). *Then there is a unique, up to isomorphism of po-groups, unital po-group* (G, u) *satisfying* (RDP_1) *such that* $(E; +, 0, 1)$ *is isomorphic with* $(\Gamma(G, u); +, 0, u)$. *Moreover, if* ϕ *is a pseudo-effect algebra isomorphism from* E *onto* $\Gamma(G, u)$, *and if* K *is any po-group with a mapping* $\psi : E \to K^+$ *which preserves* $+$ *and* 0, *then there is a unique group homomorphism* $h : G \to K$ *such that* $\psi(a) = h(\phi(a))$ *for any* $a \in E$.

3. PSEUDO MV-ALGEBRAS

MV-algebras entered mathematics by Chang (1958) in the middle of 1950s. They are a generalization of Boolean algebras to model multi-valued reasoning. We recall that according to a famous theorem of Mundici (Cignoli *et al.*, 2000), every MV-algebra $M = (M; \oplus, ^*, 0, 1)$ is an interval in an Abelian lattice-ordered group (G, u), that is, $M \cong \Gamma(G, u)$, where $0 = 0$, $1 = u$, $a \oplus b = (a + b) \wedge u$, and $a^* = u - a$ for all $a, b \in M$. Also in the Hilbert space formalism of quantum mechanics, we can meet MV-algebras which are not Boolean algebras. For example, if M is a maximal system of mutually commuting operators from $\mathcal{E}(H)$, then M can be converted into an MV-algebra (Cattaneo *et al.*, 2000). In such a case, there is a Hermitian operator $A_0 \in M$ and a system of Borel measurable functions $\{f_A : f_A : [0, 1] \to [0, 1], A \in M\}$ such that $A = f_A(A_0)$ for any $A \in M$; we define the MV-operations via $A \oplus B := (f_A \oplus f_B)(A_0)$, where $(f_A \oplus f_B)(t) := \min\{f_A(t) + f_B(t), 1\}$, $t \in [0, 1]$, and $A^* = I - A$.

We recall that according to Georgescu and Iorgulescu (2001), a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type (2, 1, 1, 0, 0) such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

If M is a pseudo MV-algebra, let the partial operation $a + b$ be defined iff $a \leq b^-$, and then $a + b := a \oplus b$. Then $(M; +, 0, 1)$ is a pseudo-effect algebra, which is a distributive lattice.

If (G, u) is a unital ℓ-group, then $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$, where $a \oplus b := (a + b) \wedge u$, $a \odot b = (a - u + b) \vee 0$, and $a^\sim = -a + u$ and $a^- = u - a$, is a pseudo MV-algebra.

We present now two examples of pseudo MV-algebras:

Example 3.1. Let $G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; +, (0, 0, 0), \leq)$ be the Scrimger 2-group, i.e.

$$(k_1, m_1, n_1) + (k_2, m_2, n_2) := \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2), & \text{if } n_2 \text{ is odd} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2), & \text{if } n_2 \text{ is even.} \end{cases}$$

Then $0 = (0, 0, 0)$ is the neutral element, and

$$-(k, m, n) = \begin{cases} (-m, -k, -n), & \text{if } n \text{ is odd} \\ s(-k, -m, -n), & \text{if } n \text{ is even,} \end{cases}$$

and G is a non-Abelian ℓ-group with the positive cone

$$G^+ = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+_{>0} \cup \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\},$$

or equivalently, $(k_1, m_1, n_1) \leq (k_2, m_2, n_2)$ iff (i) $n_1 < n_2$, or (ii) $n_1 = n_2, k_1 \leq k_2$, $m_1 \leq m_2$.

Then

$$(k_1, m_1, n_1) \vee (k_2, m_2, n_2) := \begin{cases} (k_1, m_1, n_1), & if\ n_1 > n_2 \\ (k_1 \vee k_2, m_1 \vee m_2, n_1 \vee n_2), & if\ n_1 = n_2 \\ s(k_2, m_2, n_2), & if\ n_1 < n_2, \end{cases}$$

and $u = (1, 1, 1)$ is a strong unit for G. Consequently, the corresponding pseudo MV-algebra has the form

$$\Gamma(G, u) = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\} \cup \mathbb{Z}_{\leq 1} \times \mathbb{Z}_{\leq 1} \times \{1\},$$

with

$$(k, m, 0)^- = (1 - k, 1 - m, 1),$$

$$(k, m, 0)^\sim = (1 - m, 1 - k, 1),$$

$$(k, m, 1)^- = (1 - m, 1 - k, 0),$$

$$(k, m, 1)^\sim = (1 - k, 1 - m, 0),$$

and

$$(k_1, m_1, 0) \oplus (k_2, m_2, 0) = (k_1 + k_2, m_1 + m_2, 0),$$

$$(k_1, m_1, 0) \oplus (k_2, m_2, 1) = ((m_1 + k_2) \wedge 1, (m_2 + k_1) \wedge 1, 1),$$

$$(k_1, m_1, 1) \oplus (k_2, m_2, 0) = ((k_1 + k_2) \wedge 1, (m_1 + m_2) \wedge 1, 1),$$
$$(k_1, m_1, 1) \oplus (k_2, m_2, 1) = (1, 1, 1).$$

Example 3.2. Let G be the group of all matrices of the form

$$A = \begin{pmatrix} \xi & \alpha \\ 0 & 1 \end{pmatrix},$$

where ξ and α are rational (or real) numbers such that $\xi > 0$; the group-operation is the usual multiplication of matrices. We denote $A = (\xi, \alpha)$. Then $A^{-1} = (1/\xi, -\alpha/\xi)$, and $(1, 0)$ is the neutral element. We define $G^+ := \{(\xi, \alpha) :$ where (i) $\xi > 1$, or (ii) $\xi = 1$ and $\alpha \geq 0\}$. Then G with the positive cone G^+ is a linearly ordered ℓ-group with a strong unit $U = (2, 0)$. Define $M = \Gamma(G, U)$. Then $M = M_1 \cup M_2 \cup M_3$, where $M_1 = \{(\xi, \alpha) :\ 1 < \xi < 2\}$, $M_2 = \{(2, \alpha) :\ \alpha \leq 0\}$, and $M_3 = \{(1, \alpha) :\ \alpha \geq 0\}$.

The pseudo MV-algebras of the form $\Gamma(G, u)$, where (G, u) is a unital ℓ-group, are prototypical in view of the following basic representation theorem (Dvurečenskij, 2002), which generalizes the famous result of Mundici for MV-algebras.

Theorem 3.1. *If* $(M; \oplus, ^-, ^\sim, 0, 1)$ *is a pseudo MV-algebra, then there is a unique (up to isomorphism of* ℓ*-groups) unital* ℓ*-group* (G, u) *such that* $(M; \oplus, ^-, ^\sim, 0, 1) \cong (\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$.

Moreover, there is a categorical equivalence among the category of pseudo MV-algebras (whose objects are pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras) and the category of unital ℓ-groups (whose objects are unital ℓ-groups (G, u) and morphisms are homomorphisms of unital ℓ-groups); the functor in question is given by $\Gamma :\ (G, u) \mapsto \Gamma(G, u)$. This equivalence gives a tool for a deep investigation of pseudo MV-algebras using the well-developed theory of unital ℓ-groups. We can formulate the following open problem:

Problem 1. In analogy of the Komori classification of MV-algebras, (Cignoli *et al.*, 2000), characterize all varieties of pseudo MV-algebras.

We present the following characterization of pseudo MV-algebras among pseudo-effect algebras.

Theorem 3.2. (Dvurečenskij and Vetterlein, 2001a) *Let* $(E; +, 0, 1)$ *be a pseudo-effect algebra. The following statements are equivalent:*

(i) *E is lattice-ordered and satisfies (RDP*$_0$*).*

(ii) *E satisfies (RDP$_2$).*
(iii) *E is lattice-ordered and* $a \setminus (a \wedge b) = (a \vee b) \setminus b$ *for all* $a, b \in E$.
(iv) *E is lattice-ordered and* $(E; \oplus, ^-, ^\sim, 0, 1)$ *is a pseudo MV-algebra, where* $a \oplus b := a + (a^\sim \wedge b)$, $a, b \in E$.

4. COMPATIBILITY AND BLOCKS

Orthodox quantum structures such as orthomodular latices (OML) or orthomodular posets (OMP) are not necessarily distributive structures. If they are, they are Boolean algebras. So the classical quantum structure is exactly a Boolean algebra, and it corresponds to measurements in the framework of classical mechanics. Otherwise, the quantum structures describe a measurement process in the framework of quantum mechanics which the Kolmogorovian model is not able to describe. However, in every OML or OMP it is possible to find a part which generates a Boolean subalgebra, that is, locally classical measurements. It can be expressed by compatibility of elements. It means, that two elements a and b are compatible, if there are three mutually exclusive elements a_1, b_1, c such that $a = a_1 \vee c$ and $b = b_1 \vee c$. A well-known result from quantum logic theory (Varadarajan, 1968), says that a maximal set of mutually compatible elements performs a Boolean subalgebra of a quantum logic, and, in addition, the quantum logic can be covered by blocks.

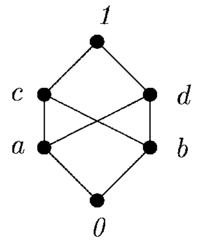
A similar result was proved by Riečanová (2000) who proved that every lattice ordered effect algebra can be covered by blocks and each such a block is an MV-subalgebra. This result can be extended also for pseudo-effect algebras, where blocks are then pseudo MV-algebras. We say that two elements a and b of a pseudo-effect algebra E are (i) *compatible* (and we write $a \leftrightarrow b$) if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c = b_1 + a_1 + c \in E$; (ii) *strongly compatible* (and we write $a \overset{\mathrm{c}}{\longleftrightarrow} b$) if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, $a_1 + b_1 + c = b_1 + a_1 + c \in E$, and $a_1 \wedge b_1 = 0$, (iii) *weakly compatible*, (and we write $a \overset{\mathrm{w}}{\longleftrightarrow} b$) if there exist three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c \in E$ and $b_1 + a_1 + c \in E$. It is evident that (ii) implies (i) and (i) implies (iii).

If $a \leq b$, then $a \overset{\mathrm{c}}{\longleftrightarrow} b$ (set $a_1 = 0$, $b_1 = b \setminus a$, $c = a$).

For example, if M is a pseudo MV-algebra, then all elements are strongly compatible. We note that if E is a lattice, and $a \overset{\mathrm{c}}{\longleftrightarrow} b$, then elements in the corresponding decompositions are uniquely determined. Namely, $c = a \wedge b$, $a_1 = a \setminus (a \wedge b)$ and $b_1 = b \setminus (a \wedge b)$. Moreover, if E is a lattice-ordered pseudo-effect algebra, then all three kinds of compatibilities coincide (Dvurečenskij and Vetterlein, 2003). On the other hand, in (Pulmannová, 2002) it was shown that for $\mathcal{E}(H)$, compatibility and strong compatibility coincide, however, $\mathcal{E}(H)$ is not a lattice (see Dvurečenskij and Pulmannová, 2000).

Example 4.1. Let $E = \{0, a, b, c, d, 1\}$, where the addition $+$ is defined in the table.

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	1	$\times$	$\times$
b	b	c	d	$\times$	1	$\times$
c	c	1	$\times$	$\times$	$\times$	$\times$
d	d	$\times$	1	$\times$	$\times$	$\times$
1	1	$\times$	$\times$	$\times$	$\times$	$\times$
$'$	0	a	b	c	d	1
	1	c	d	a	b	0



Then E is an effect algebra which is not a lattice and does not fulfill (RIP), but all elements of E are strongly compatible and e.g. $c \overset{\text{c}}{\longleftrightarrow} d$ and $c \vee d \in E$ but $c \wedge d \notin E$ as well as $a \overset{\text{c}}{\longleftrightarrow} b$, $a \wedge b \in E$ but $a \vee b \notin E$.

It is possible to prove the following statement from (Dvurečenskij and Vetterlein, 2003).

Proposition 1. *Let E be a lattice pseudo-effect algebra, let $a_i \leftrightarrow b$ for any $i \in I$, and $a := \bigvee_{i \in I} a_i \in E$. Then $b \leftrightarrow a$ and*

$$\bigvee_i (a_i \wedge b) = \left(\bigvee_i a_i \right) \wedge b.$$

Let $\{E_t\}_{t \in T}$ be a system of pseudo-effect algebras such that $E_t \cap E_s = \{0, 1\}$ for $t \neq s$. The set $E := \bigcup_{t \in T} E_t$ can be organized into a pseudo-effect algebra such that $x + y$ is defined in E iff $x, y \in E_t$ for some $t \in T$ and if $x + y$ is defined in E_t, and in such a case, $x + y$ takes the value from E_t. Then E is a pseudo-effect algebra, which is said to be a *horizontal sum* of the system of pseudo-effect algebras $\{E_t\}_{t \in T}$.

A maximal set of mutually compatible elements of a pseudo-effect algebra E is said to be a *block*.

For example, if E is a pseudo MV-algebra, then E is a unique block in E. In addition, if E is a horizontal sum of a system of pseudo MV-algebras $\{E_t\}_{t \in T}$, then E is not necessarily a pseudo MV-algebra, and $\{E_t\}_{t \in T}$ is the system of all blocks in E.

Riečanová has proved that every block in a lattice-ordered effect algebra is an MV-algebra. However, this is not true if E is a lattice-ordered pseudo-effect algebra, as it was shown in (Dvurečenskij and Vetterlein, 2003). Therefore, we need the following notion: We say that a pseudo-effect algebra E has the *difference compatibility property* (DCP) for short, if $a \leftrightarrow b$, $a \leftrightarrow c$ and $b \leq c$ imply $c \leftrightarrow c \setminus b$. Every pseudo MV-algebra, or every horizontal sum of pseudo MV-algebras, or every effect algebra, or any horizontal sum of the previous algebras has (DCP).

This property for lattice-ordered pseudo-effect algebras is equivalent with the following notion: pseudo-effect algebra E satisfies the *compatibility complement property* (CCP) for short, if $a \leftrightarrow b$ implies $a \leftrightarrow 1 \setminus b$; then also $a \leftrightarrow b / 1$.

We can now present the main result of the present section.

Theorem 4.1. *Let E be a lattice-ordered pseudo-effect algebra with (DCP). Then every block of E is a pseudo-effect subalgebra of E which is a pseudo MV-algebra. Moreover, any such pseudo-effect algebra E is a set-theoretical union of its blocks.*

5. STATES ON PSEUDO-EFFECT ALGEBRAS AND PSEUDO MV-ALGEBRAS

A state on a propositional system is connected with the intent of capturing the notion of "average degree of truth" of a proposition, and it goes back to Boole's ideas (1967).

A *state* on a pseudo-effect algebra E is any mapping $m : E \to [0, 1]$ such that $m(1) = 1$, and $m(a + b) = m(a) + m(b)$ whenever $a + b$ is defined in E. A *state* on a unital po-group (G, u) is a mapping $s : G \to \mathbb{R}$ such that (i) $s(g_1 + g_2) = s(g_1) + s(g_2)$ for all $g_1, g_2 \in G$, (ii) $s(g) \geq 0$ for any $g \geq 0$, and (iii) $s(u) = 1$. If s is a state on (G, u), then $m := s|\Gamma(G, u)$ is a state on $\Gamma(G, u)$. Conversely, if m is a state on $\Gamma(G, u)$ for a unital po-group (G, u) satisfying (RDP$_1$), then by Theorem 2.1, m can be uniquely extended to a state s on (G, u) such that $m = s|\Gamma(G, u)$. Such a situation happens, for example, if E is a pseudo MV-algebra. Denote by $\mathcal{S}(E)$ and $Ext(\mathcal{S}(E))$ the set of all states on E and the set of all extremal states on E. We note that if E is an interval effect algebra, then it possesses at least one state. But even if M is a pseudo MV-algebra, then it can happen that M possesses no state (Dvurečenskij, 2001). We have more information on the state space for pseudo MV-algebras.

Let m be a state on a pseudo MV-algebra M, denote by $\mathrm{Ker}(m) := \{a \in M : m(a) = 0\}$. Then $\mathrm{Ker}(m)$ is a normal ideal[4] of M, and on the quotient $M/\mathrm{Ker}(m)$ we have that it is always an MV-algebra. In addition, $\tilde{m}(a/\mathrm{Ker}(m)) := m(a)$ $(a \in M)$ is a state on the MV-algebra $M/\mathrm{Ker}(m)$.

[4] A non-void subset I of M is said to be an *ideal* of pseudo MV-algebra M if (i) $x \oplus y \in I$ whenever $x, y \in I$, and (ii) if $x \leq y$, $x \in M$ and $y \in I$, then $x \in I$. An ideal I is said to be *normal* if $x \oplus I = I \oplus x$ for any $x \in M$.

On the other hand, we can describe the set of all extremal states on M as follows; for more details see (Dvurečenskij, 2001). Let m be an extremal state on M. Then Ker(m) is always a normal maximal ideal of M, and conversely, for any normal maximal ideal I there exists a unique extremal state m on M such that $I = \text{Ker}(m)$. Moreover, a state m is extremal iff $m(a \wedge b) = \min\{m(a), m(b)\}$ for all $a, b \in M$. It is possible to show that any linearly ordered pseudo MV-algebra possesses a (unique) state as well as any pseudo MV-algebra which is a subdirect product of linearly ordered pseudo MV-algebras. Even every normal valued pseudo MV-algebra (for definition see Dvurečenskij, 2001) possesses at least one state.

For example, take $M = \Gamma(G, u)$ from Example 3.1, and define a mapping m on M by $m((k, n, 0)) = 0$ and $m((k, n, 1)) = 1$. Then m is a unique state on this pseudo MV-algebra M which is not an MV-algebra (this is a normal valued pseudo MV-algebra). In Example 3.2, M_3 is a unique normal and maximal ideal of M, and there is a unique state-morphism m, namely $m((\xi, \alpha)) = \log_2(\xi)$, $(\xi, \alpha) \in M$.

If now E is a pseudo-effect algebra, then the investigation of the state space of E is more complicated than that for pseudo MV-algebras. The problem is that it is not easy to define ideals and quotient pseudo-effect algebras. The basic properties of such states are given in (Dvurečenskij and Vetterlein, 2001c). We can only add that if E possesses at least one state, then sometimes (for example if E is a pseudo MV-algebra) it is possible to have an effect algebra E_0 which is a quotient of the pseudo-effect algebra E, i.e., $E_0 = E/I_0$, where I_0 is an appropriate ideal of E such that each induced state $\tilde{s}$ from E_0 has the same value as s on E, i.e. $\tilde{s}(a/I_0) = s(a)$ $(a \in E)$ for any state s on E. In other words, if E is an interval pseudo-effect algebra with a state s, then the non-commutative structure of the original propositional system E can be eliminated by finding a (commutative) effect algebra such that the statistical information remains.

6. CONCLUSION

We have presented elements of pseudo-effect algebras and of pseudo MV-algebras. They are a generalization of effect algebras and MV-algebras by no longer assuming the commutativity. We have shown how the algebraic structure of pseudo-effect algebras implies a whole unital po-group. This is possible for example if E satisfies a kind of the Riesz decomposition property. This Riesz-type decomposition property is roughly speaking another kind of distributivity; it is connected with refinements of decompositions of the unity. Every pseudo MV-algebra is of this kind, and it is an interval in a unital ℓ-group.

The partial operation $+$ is connected with a state structure of propositions, and it corresponds to original ideas of Boole (1967), who said that for statistical information of the propositional system is only necessary to have a rule how we can add probabilities of events.

If we have a fixed pseudo-effect algebra E, we have shown when it is possible to cover it by blocks. In such a case, a block is a maximal set of mutually compatible elements of E which correspond to "classical" subpart of the propositional system. In contrast to orthodox quantum structures, such a block is a pseudo MV-algebra, and pseudo MV-algebras are analogues of Boolean algebras.

The existence of states on pseudo-effect algebras gives an opportunity for performing a measuring process. As an important consequence of such a possibility is the fact that then we can substitute the whole statistical information involved in E by the same statistical information concentrated on (commutative) effect algebras. So we are able to find an ideal, i.e. a appropriate filter, which kills the non-commutativity and which preserves a statistical information of the original statistical information which can be influenced by a noise causing the non-commutativity of the propositional system.

In the present, a pure quantum physical experiment which can be described by non-commutative pseudo-effect algebras is not yet known, and it would be interesting to have it. In any rate we have suggested an algebraical system which describe a special kind of non-commutativity and which was inspired by our experience with quantum structures.

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